

Symmetry and Lie-Frobenius reduction of differential equations

G. Gaeta*

*Dipartimento di Matematica, Università degli Studi di Milano,
via C. Saldini 50, 20133 Milano (Italy)*
giuseppe.gaeta@unimi.it

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Abstract

Twisted symmetries, widely studied in the last decade, proved to be as effective as standard ones in the analysis and reduction of nonlinear equations. We explain this effectiveness in terms of a Lie-Frobenius reduction; this requires to focus not just on the prolonged (symmetry) vector fields but on the distributions spanned by these and on systems of vector fields in involution in Frobenius sense, not necessarily spanning a Lie algebra.

Introduction and motivation

Nonlinear systems are relevant in Mathematics and Physics, but they are as a rule hard to analyze. One of the key tools in tackling nonlinear differential equations (by this we always mean possibly a system) is provided by symmetry analysis [1, 14, 35, 51, 52, 60], and actually this tool – and more generally the theory of Lie groups – was created by S. Lie precisely to study nonlinear differential equations.

The theory received a sound geometrical setting thanks to the work of Cartan and Ehresmann, with the theory of Jet bundles. It received a new boost several decades after the work of Lie and Cartan, when the work of Ovsjannikov in the USSR and by Birkhoff in the USA [4, 55] revived interest in it. Applications of the theory beyond the simplest case require rather extensive computations, but these are nowadays standard thanks to computer algebraic manipulation languages [31, 32, 33, 57].

The renewal of interest also called for generalizations of the theory; thus different kind of symmetries extending the classical concept were considered in the literature. In all of these cases, one considers vector fields in the extended phase manifold (which later on in this paper will be denoted as M) and their

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standard prolongation to the relevant Jet bundle $J^n M$; generalization consists either in considering vector fields more general than Lie-point ones (e.g. generalized vector fields [35, 51]), or in weakening the requirements regarding their action on the set of solutions of the equation under study (e.g. requiring that there are some invariant solutions albeit the vector fields do not fully qualify as symmetries, as in conditional or nonclassical symmetries [21, 36, 37]; or that only a subset of solutions is invariant under their action, as in partial symmetries [15]). The fact that one always considers standard prolongation is entirely natural, as once the transformation of independent and dependent variables are assigned, the transformation laws for derivatives of the latter with respect to the former are also entirely determined: this is precisely the content of the standard prolongation formula [1, 14, 35, 51, 52, 60].

In recent years, starting with the work of Muriel and Romero in 2001, it was realized that one can extend the theory in a different, and somehow surprising, direction. That is, it was realized that one could consider a deformation, or a “twisting”, of the prolongation operation, having twisted prolongations of Lie-point vector fields (or of more general types of vector fields, but here we will only focus on the Lie-point case). When these twisted prolongations satisfy the usual relation with (the geometrical object in $J^n M$ representing) the differential equations under study, one speaks of *twisted symmetries*. The twisting is always based on an auxiliary object (a function, a matrix, a one-form, depending on the type of twisted symmetry one is considering; see below for details); when the latter vanishes, one is reduced to standard prolongations and symmetries; thus, twisted symmetries represent a genuine extension of standard ones.

The nontrivial fact is that these twisted symmetries turn out to be *as effective as standard ones* in the analysis of differential equations. This applies both to ODEs, where one looks for a symmetry reduction of the system under study, and to PDEs, where one looks for invariant solutions. In fact, as twisted symmetries are more general than standard ones, there are cases where we have no standard symmetry but there are twisted symmetries (as was the case in the pioneering study of Muriel and Romero on λ -symmetries [40, 41]), so that equations which cannot be reduced or integrated within the framework of standard symmetries turn out to be integrated via the standard procedure if one resorts to twisted symmetries. In other words, twisted symmetries are an extra tool to study nonlinear equations; and one which can work where standard symmetries fail.

The purpose of this paper is to understand *why* twisted symmetries are as effective as standard ones. In order to do this, we will slightly change the usual focus in determining the symmetry properties of differential equations (with a formulation fully equivalent to the standard one), looking – even in the case of a single vector field – at *distributions* rather than at vector fields. This will immediately call for consideration of possible changes of the set of generators for the distribution, and we will find that such a change leads to considering twisted prolongations and symmetries.

In view of the dominant role assumed by distributions, and by the involution rather than the Lie algebraic structure, we will speak of *Lie-Frobenius reduction* to characterize our approach.

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1 General framework

Let us recall the general framework for the symmetry analysis of differential equations. Our task here is mainly to fix notation, and we will assume the reader has some familiarity with the subject; see e.g. [1, 14, 35, 51, 52, 60]. All the objects considered (manifolds, functions, etc.) will be assumed to be smooth, i.e. of class C^∞ , and real. We will use the Einstein convention for sum on repeated indices and multi-indices.

1.1 Notations for derivatives

First of all, a word about compact notation for partial derivatives. In the presence of coordinates x^i and u^a , we will write

$$\partial_i := \frac{\partial}{\partial x^i}, \quad \partial_a := \frac{\partial}{\partial u^a}.$$

We will freely use the multi-index notation for partial derivatives; thus while first order partial derivatives of $u^a = u^a(x^1, \dots, x^q)$ will be denoted by $u_i^a := (\partial u^a / \partial x^i)$, for higher order derivatives we will consider multi-indices $J = (j_1, \dots, j_q)$, with $j_i \in \mathbb{N}$, of order $|J| = q_1 + \dots + q_n$, and write

$$u_J^a := \frac{\partial^{|J|} u^a}{\partial (x^1)^{j_1} \dots \partial (x^q)^{j_q}}.$$

The notation $\hat{J} = \{J, i\}$ will denote the multi-index with entries $\hat{j}_k = j_k + \delta_{ik}$, and correspondingly $u_{J,i}^a := (\partial u_J^a / \partial x^i)$. We will also need to consider derivatives with respect to u_J^a ; for these we will also use the notation

$$\partial_a^J := \frac{\partial}{\partial u_J^a}.$$

We will denote the set of derivatives (of the u with respect to the x) of order n as $u^{[n]}$, that of derivatives of order from 0 to n (included) as $u^{(n)}$; thus $f(x, u^{(n)})$ will denote a function of x , u and of derivatives of order up to n .

Finally, we will consider total derivatives D_i with respect to the independent variables x^i ; in this notation, these are also written as

$$D_i = \partial_i + u_{J,i}^a \partial_a^J.$$

1.2 Equations and symmetries

We can now pass to describe our general framework (see again [1, 14, 35, 51, 52, 60] for details). Consider a bundle (M, π, B) with fiber $\pi^{-1}(x) = U$; we will use local coordinates (x^1, \dots, x^q) on the base manifold B , and (u^1, \dots, u^p) on the fiber U ; a function $u = f(x)$ corresponds to a section γ_f of (M, π, B) . Associated to (M, π, B) are the Jet bundles $J^n M$; natural local coordinates on these are provided by (x, u) and partial derivatives (up to order n) of the u with respect to the x . The Jet bundles are naturally endowed with a *contact structure* \mathcal{C} ; this is generated by the *contact forms*, given in local coordinates by

$$\omega_J^a := du_J^a - u_{J,i}^a dx^i .$$

A function $u = f(x)$ also identifies its partial derivatives of any order; in the same way, a section γ_f of (M, π, B) also identifies a section $\gamma_f^{(n)}$ of $(J^n M, \pi_n, B)$, i.e. of $J^n M$ seen as a bundle over B . The section $\gamma_f^{(n)} \in \Sigma(J^n M)$ is said to be the *prolongation* of $\gamma_f \in \Sigma(M)$.

A differential equation, or system thereof, Δ (of order n) is a relation involving the independent variables x , the dependent variables u , and their derivatives (up to order n); thus it defines a *solution manifold* S_Δ in $J^n M$. The function $u = f(x)$ is a solution to Δ if and only if the prolongation $\gamma_f^{(n)}$ of γ_f lies entirely in, i.e. is a submanifold of, S_Δ .

Let us now consider a vector field X in M ; this is written in coordinates as

$$X = \xi^i(x, u) \partial_i + \varphi^a(x, u) \partial_a . \quad (1)$$

This generates a (local) one-parameter group of diffeomorphisms in M ; its action on $\Sigma(M)$ is described by its *evolutionary representative*

$$X_v = (\varphi^a - u_i^a \xi^i) \partial_a := Q^a(x, u, u_x) \partial_a . \quad (2)$$

That is, the infinitesimal action of X maps the function $u = f(x)$ into the function $u = \hat{f}(x)$, with $\hat{f}(x) = f(x) + \varepsilon[\varphi^a(x, u) - u_i^a \xi^i(x, u)]$, where u and u_i should be computed on $u = f(x)$.

The action of X on M induces an action on $J^n M$; this is described by the vector field

$$X^{(n)} = \xi^i(x, u) \partial_i + \psi_J^a(x, u, \dots, u^{(|J|)}) \partial_a^J \quad (3)$$

where the coefficients ψ_J^a depend on partial derivatives u_K^a of order up to $|J|$ and satisfy, with $\psi_0^a \equiv \varphi^a$, the *prolongation formula*

$$\psi_{J,i}^a = D_i \psi_J^a - u_{J,k}^a D_i \xi^k . \quad (4)$$

The vector field X is said to be a *symmetry generator*, or (with a slight abuse of language) simply a *symmetry* for Δ if it (locally, i.e. near to zero in the group parameter) maps solutions into solutions; this is seen [1, 14, 35, 51, 52, 60] to be equivalent to the following

Definition 1. The vector field X in M is a symmetry generator for the differential equation Δ of order n if and only if $X^{(n)} : S_\Delta \rightarrow TS_\Delta$.

Remark 1. Note that if $Y = cX$ with c a constant, then $Y^{(n)} = cX^{(n)}$. Thus if X is a symmetry of Δ , so is $Y = cX$ for any $c \in \mathbf{R}$.

On the other hand, if Y is collinear but not proportional to X , i.e. $Y = f(x, u)X$ with f non constant, then their prolongations do *not* satisfy the same relations, and are in general not collinear. This is readily seen considering first prolongations, which is enough to make our point. In fact, for X given by (1), we have $Y = \eta^i \partial_i + \beta^a \partial_a$, with $\eta^i = f \xi^i$ and $\beta^a = f \varphi^a$. The prolongation formula (4) provides then $X^{(1)} = X + \psi_i^a \partial_a^i$ and $Y^{(1)} = Y + \chi_i^a \partial_a^i$ with $\psi_i^a = D_i \varphi^a - u_k^a D_i \xi^k$ and

$$\begin{aligned} \chi_i^a &= D_i \beta^a - u_k^a D_i \eta^k = D_i (f \varphi^a) - u_k^a D_i (f \xi^k) \\ &= f (D_i \varphi^a - u_k^a D_i \xi^k) + (\varphi^a - u_k^a \xi^k) (D_i f) . \end{aligned}$$

Thus, writing $Q^a = (\varphi^a - u_k^a \xi^k)$, we have

$$Y^{(1)} = f X^{(1)} + Q^a (D_i f) \partial_a^i = f \left[X^{(1)} + f^{-1} (D_i f) Q^a \partial_a^i \right] .$$

For higher order prolongations, we get a similar result; see the Appendix A. \odot

1.3 Sets of vector fields

Let us now consider a set of vector fields X_α , written in coordinates as

$$X_\alpha = \xi_\alpha^i(x, u) \partial_i + \varphi_\alpha^a(x, u) \partial_a \quad (5)$$

and with prolongations

$$X_\alpha^{(n)} = \xi_\alpha^i(x, u) \partial_i + \psi_{\alpha; J}^a(x, u, \dots, u^{(|J|)}) \partial_a^J . \quad (6)$$

We are interested in the case where the X_α are in *involution* (in Frobenius sense), i.e. the set is closed under commutation:

$$[X_\alpha, X_\beta] = F_{\alpha\beta}^\gamma(x, u) X_\gamma , \quad (7)$$

with $F_{\alpha\beta}^\gamma$ smooth functions on M . If the $F_{\alpha\beta}^\gamma(x, u)$ are actually constant (we will then write them as $c_{\alpha\beta}^\gamma$) we have a *Lie algebra* of vector fields.

It is well known [1, 14, 35, 51, 52, 60] that if X, Y are vector fields on M , and $X^{(n)}, Y^{(n)}$ their prolongations, then

$$[X^{(n)}, Y^{(n)}] = ([X, Y])^{(n)} . \quad (8)$$

In a slightly different formulation,

$$[X, Y] = Z \Rightarrow [X^{(n)}, Y^{(n)}] = Z^{(n)} .$$

It follows easily from this that if the vector fields X_α span a Lie algebra, then their prolongations span the same Lie algebra; in fact,

$$\left[X_\alpha^{(n)}, X_\beta^{(n)} \right] = ([X_\alpha, X_\beta])^{(n)} = \left(c_{\alpha\beta}^\gamma X_\gamma \right)^{(n)} = c_{\alpha\beta}^\gamma X_\gamma^{(n)}.$$

Note this depends on the $c_{\alpha\beta}^\gamma$ being constant.¹

Remark 2. On the other hand, if the X_α are in involution but do *not* span a Lie algebra (that is, at least some of the $F_{\alpha\beta}^\gamma$ are not constant), then their prolongations do *not* satisfy the same involution relations, and could very well not be in involution. More precisely, we have that if the X_α satisfy eq. (7), then

$$\left[X_\alpha^{(n)}, X_\beta^{(n)} \right] = ([X_\alpha, X_\beta])^{(n)} + Z_{\alpha\beta}^{(n)}, \quad (9)$$

where vector field $Z_{\alpha\beta}^{(n)}$ is nonzero; the recursion relation obeyed by its coefficients will be determined in the Appendix A. \odot

Remark 3. Let X_α be symmetries of Δ . It is clear that for any choice of smooth functions $f_\alpha : J^n M \rightarrow \mathbf{R}$, the vector field $Z = f_\alpha X_\alpha^{(n)}$ is tangent to S_Δ . On the other hand, a generic vector field in $J^n M$ of this form, for f non constant and generic, will *not* be the prolongation of *any* vector field in M , and surely not of the vector field $Z_0 = f_\alpha X_\alpha$. Note that – as we will discuss later on – there will be special choices of non-constant f_α for which Z is a prolongation of some vector field $W \neq Z_0$ in M . \odot

2 An equivalent formulation of the symmetry condition

The discussion of the previous section suggests to shift focus in our description of symmetries (keeping of course the same meaning); that is, rather than focusing on a vector field X , or set of vector fields $\mathcal{X} = \{X_1, \dots, X_r\}$, we will look at the **distribution** \mathcal{D}_X or $\mathcal{D}_\mathcal{X}$ their prolongation span in $TJ^n M$, also called the *prolonged distribution*, or more precisely at the integral manifolds for these distributions [29].

In the case of a single vector field, \mathcal{D}_X is one-dimensional (note it may have singular points, corresponding to singular points – i.e. zeros – of X); for a set \mathcal{X} , the dimension of $\mathcal{D}_\mathcal{X}$ corresponds to the rank of \mathcal{X} and again we can have singular points where $\mathcal{D}_\mathcal{X}$ has a lower dimension.

Definition 2. *The vector field X in M is a symmetry generator for the differential equation Δ of order n if $S_\Delta \subset J^n M$ is an integral manifold for the*

¹We recall in passing that, as well known, the symmetry generators of a given Δ form a Lie algebra under commutation. Note that this could be infinite-dimensional as a Lie algebra (albeit spanning a finite-dimensional distribution, as guaranteed by the finite dimensionality of M and of $J^n M$).

distribution \mathcal{D}_X spanned by $X^{(n)}$ in $J^n M$. The set $\mathcal{X} = \{X_1, \dots, X_r\}$ of vector fields in M is a set of symmetry generators for the differential equation Δ of order n if $S_\Delta \subset J^n M$ is an integral manifold for the distribution $\mathcal{D}_\mathcal{X}$ spanned by $X_\alpha^{(n)}$ in $J^n M$.

This definition is equivalent to the previous one, but shifts our attention from the vector fields to the distribution they generate.

It is quite obvious that if we drop the requirement to have prolonged vector fields we have many different sets of generators for \mathcal{D}_X or $\mathcal{D}_\mathcal{X}$; but in this case the symmetry is, in general, of little use for the usual goals of symmetry analysis; that is, for reducing the equation under study (in the case of ODEs), or at least for determining its invariant solutions (for PDEs).

We thus wonder if there is some intermediate class: vector fields in $J^n M$ which are not prolongations of vector fields in M , but which can still be used for symmetry reduction. As mentioned above, one should distinguish between ODEs and PDEs

(1) Let us first consider ODEs. Looking at the standard symmetry reduction procedure for ODEs [1, 14, 35, 51, 52, 60], we realize this is based on the so called *Invariant By Differentiation Property* (IBDP). This means that once we know differential invariants of order zero (call them η) and one (call them $\zeta^{(1)}$) for a vector field, differential invariants of order two can be computed simply by

$$\zeta^{(2)} = \frac{D_x \zeta^{(1)}}{D_x \eta} ;$$

the procedure is then iterated to generate higher order invariants. Note in this way we generate (for ODEs, i.e. for a single independent variable x) a complete set of differential invariants of any given order.

It should be remarked that differential invariants will be the same for the vector field $Y = X^{(n)}$ and for $Z = fY$, for any smooth function f which is nowhere vanishing, or at least which vanishes only at singular points of Y . This shows immediately that the IBDP property will hold true for any vector field of the form $Z = fY$, i.e. that (as it could be expected) symmetry reduction is still possible if we consider a different generator for the one-dimensional distribution generated by $Y = X^{(n)}$ in $J^n M$.

It is then natural to ask if this vector field $Z = fY$ can be associated in some way to a vector field W in M , through some kind of deformed prolongation operation.

Similar considerations hold for sets of vector fields, i.e. for $Y_\alpha = X_\alpha^{(n)}$ and $Z_\alpha = F_\alpha{}^\beta Y_\beta$ with F a smooth regular matrix function.

(2) As for determining invariant solutions to PDEs (by this we mean solutions which are invariant under some subalgebra \mathcal{G}_0 of the full symmetry algebra \mathcal{G} of the equation; one could actually also consider more general settings [21, 36, 15], but here we will not consider these), this is based on the restriction of the equation to the set of functions which are invariant under the symmetry

vector field. The restriction of this equation corresponds, in geometrical terms, to the intersection of the solution manifold S_Δ with the fixed point set \mathcal{F}_0 for all the vector fields $Y_\alpha = X_\alpha^{(n)}$ given by prolongations of the generators $\mathcal{X} = \{X_\alpha\}$ of the invariance subalgebra \mathcal{G}_0 . From our point of view \mathcal{F}_0 is the set of maximally singular points for the distribution $\mathcal{D}_\mathcal{X}$. Again, it is clear that changing the system of generators by passing to $Z_\alpha = F_\alpha^\beta Y_\beta$ for this distribution does not change \mathcal{F}_0 , and hence the invariant solutions $S_\Delta \cap \mathcal{F}_0$.

In this context too it is natural to ask if these vector field $Z_\alpha = F_\alpha^\beta Y_\beta$ can be associated in some way to vector fields W_α in M , through some kind of deformed prolongation operation.

The question raised in points (1) and (2) above appears to be new, but it has actually been implicitly solved (working from a different point of view and with a different focus) in the literature dealing with *twisted prolongations* of vector fields and *twisted prolongations*.

Remark 4. The field of twisted symmetries of differential equations was started by the works of Muriel & Romero with the so called λ -symmetries [40], and by the geometrical understanding of these by Pucci & Saccomandi [56]; the scope of the theory, initially limited to scalar ODEs, was then extended to systems of ODEs [43] and to PDEs [16, 28], and recently also to sets of vector fields rather than single ones [17, 18, 19, 20]. See [25, 27] for reviews on twisted prolongations and twisted symmetries. Roughly speaking, one consider a modification (twisting) of the standard prolongation rule, depending on an auxiliary object², but such that the vector fields whose such twisted prolongations is tangent to S_Δ can still be used in the same way as standard symmetries for reducing the differential equations under study (in the case of ODEs or systems thereof) or to obtain invariant solutions (in the case of PDEs or systems thereof). The fact that twisted symmetries are as useful as standard ones for these tasks is due to certain algebraic facts (for ODEs and reduction, to satisfying the Invariant By Differentiation Property (IBDP) [42]; for PDEs, to the fact twisted prolongations coincide with standard ones on the set of invariant functions [16, 28]); we trust the present work sheds light on the geometrical reason behind these. \odot

Remark 5. It should be stressed that "twisted symmetry" is in a way a misnomer: a twisted symmetry does in general *not* map solutions into solutions, and hence is not, properly speaking, a symmetry [1, 14, 35, 51, 52, 60] (but for PDEs, see point (2) above, when they can be applied to reduce the equation they are at least conditional [36], or partial [15], symmetries). Moreover while standard symmetries can be computed algorithmically, this is not the case for twisted symmetries: their determination relies in general on guesswork or physical considerations, albeit in some cases the analysis of the system under study can provide insights for the structure of twisted symmetries [18]. \odot

²This may be a scalar function $\lambda(x, u, u_x)$ in λ -symmetries, a set of q ($p \times p$) matrix functions $\Lambda_i(x, u^{(1)})$ defining the semi-basic one-form $\mu = \Lambda_i dx^i$ satisfying the horizontal Maurer-Cartan equation as in μ -symmetries, or a matrix-valued function $\sigma^{\alpha\beta}(x, u^{(1)})$ (here the indices refer to a generating set of vector fields [17, 18]) as in σ -symmetries.

3 Distributions and twisted symmetries

As mentioned above, we will use some results from investigations on twisted symmetries; we will quote these from the review paper [27]. It turns out the results are more simply stated in terms of evolutionary representatives of vector fields, see (2) above. We will find that changing the set of generators in the prolonged distributions associated to a vector field or a set of vector fields leads quite naturally to consider twisted prolongations and symmetries.

3.1 Single vector fields

We start by considering a single vector field X , which we write in the form (1). In this context, we consider its μ -prolongation, based on a semi-basic one form μ on (J^1M, π_1, B) , which we write as

$$\mu = \Lambda_i(x, u^{(1)}) dx^i ;$$

the Λ_i are $(p \times p)$ matrices (recall p is the dimension of the fiber $\pi^{-1}(x)$ in M), satisfying the compatibility conditions

$$D_i \Lambda_j - D_j \Lambda_i + [\Lambda_i, \Lambda_j] = 0 . \quad (10)$$

The latter is just the horizontal Maurer-Cartan equation [11, 58]. (Dealing with the horizontal version of this is rather natural in view of the presence of the contact structure in J^nM .) By introducing the operators

$$\nabla_i := D_i + \Lambda_i ,$$

defined more precisely as $(\nabla_i)^a_b = (\delta^a_b)D_i + (\Lambda_i)^a_b$, this is also reformulated as the zero-curvature condition [10, 22, 38]

$$[\nabla_i, \nabla_j] = 0 .$$

The μ -prolongation of the vector field X is defined as the vector field (3) with coefficients ψ_J^a satisfying the *μ -prolongation formula*

$$\psi_{J,i}^a = D_i \psi_J^a - u_{J,k}^a D_i \xi^k + (\Lambda_i)^a_b (\psi_J^b - u_k^b \xi^k) ; \quad (11)$$

this is also rewritten in terms of the ∇_i defined above as

$$\psi_{J,i}^a = (\nabla_i)^a_b \psi_J^b - u_{J,k}^b (\nabla_i)^a_b \xi^k .$$

In order to stress the application of the modified prolongation operation, we will denote the n -th μ -prolongation of X as $X_\mu^{(n)}$.

Remark 6. Note that for ODEs we have only one matrix Λ ; a special case occurs, as discussed below, when Λ is a multiple of the identity via a smooth function $\lambda : J^1M \rightarrow R$, i.e. for $\Lambda = \lambda I$. In this case we are reduced to the setting of λ -prolongations and symmetries introduced by Muriel and Romero

for scalar equations [40, 41, 42] and for systems[43]. Thus λ -prolongations and symmetries can be seen, for the sake of our present discussion, as special cases of μ -prolongations and symmetries. (The same would apply for σ -prolongations and symmetries to be considered below; in that case we obtain λ ones when the set reduces to a single vector field and hence the matrix σ to a scalar function λ .) However, this special case has some special – and quite convenient – features, see below. \odot

Lemma 1. *Consider the evolutionary vector fields X and \tilde{X} on M given by $X = Q^a \partial_a$ and $\tilde{X} = (A^a_b Q^b) \partial_a$, with $A : M \rightarrow \mathbf{Mat}(\mathbf{R}, q)$ a nowhere zero smooth (C^∞) matrix function on M . Consider moreover the standard prolongation of \tilde{X} and the μ -prolongation of X with $\mu = A^{-1}(DA)$; i.e. the vector fields $X^{(n)}_\mu = Q^a_J \partial_a^J$ and $\tilde{X}^{(n)} = \tilde{Q}^a_J \partial_a^J$ in $J^n M$. Then*

$$A^a_b Q^b_J = \tilde{Q}^a_J. \quad (12)$$

Proof. First of all we note that by definition the Q^a_J obey the μ -prolongation formula, and the \tilde{Q}^a_J the standard one, so that

$$Q^a_{J,i} = D_i Q^a_J + (\Lambda_i)^a_b Q^b_J, \quad \tilde{Q}^a_{J,i} = D_i \tilde{Q}^a_J.$$

Let us write, for all J , $Q^a_J = L^a_b P^b_J$, with L a C^∞ and nowhere singular matrix function on M . Then the μ -prolongation formula requires

$$\begin{aligned} Q^a_{J,i} &= D_i(Q^a_J) + (\Lambda_i)^a_b Q^b_J = D_i(L^a_b P^b_J) + (\Lambda_i)^a_b L^b_c P^c_J \\ &= D_i(L^a_b) P^b_J + L^a_b D_i(P^b_J) + (\Lambda_i L)^a_b P^b_J. \end{aligned}$$

On the other hand, we know that $Q^a_{J,i} = L^a_b P^b_{J,i}$; comparing these two formulas, we get

$$P^a_{J,i} = D_i P^a_J + [L^{-1} D_i(L)]^a_b P^b_J + (L^{-1} \Lambda_i L)^a_b P^b_J.$$

We conclude that the P^a_J satisfy the standard prolongation formula – and thus can be identified with the \tilde{Q}^a_J – provided L satisfies $L^{-1}(D_i L) + L^{-1} \Lambda_i L = 0$; equivalently, provided $(D_i L)L^{-1} = -\Lambda_i$. Writing this relation in terms of $A = L^{-1}$, we have

$$\Lambda_i = A^{-1} (D_i A). \quad (13)$$

As $DA = (D_i A) dx^i$, the proof is completed. \triangle

Remark 7. We can summarize the relations described by this Lemma in the form of a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{A} & \tilde{X} \\ \downarrow \mu\text{-}prol & & \downarrow prol \\ X^{(n)}_\mu & \xrightarrow{A} & \tilde{X}^{(n)}_0 \end{array} \quad (14)$$

The matrices Λ_i defining μ and A are related by (13). The C^∞ smoothness of A entails C^∞ smoothness of the Λ_i . In the case $\Lambda_i = \lambda_i I$, the equivalence is through a simple rescaling of the vector fields. The case of λ -symmetries is a special case of that of μ -symmetries. \odot

Remark 8. Needless to say, the diagram can also be read the other way round. That is, considering the map $L = A^{-1}$, we state that acting with the map L on $\tilde{X}_0^{(n)}$ (and on \tilde{X}) we obtain a vector field $L \circ \tilde{X}_0^{(n)}$ which is the λ -prolongation of the vector field $X = L \circ \tilde{X}$. \odot

Remark 9. Here the matrix A acts on the vector indices corresponding to the variables of the space tangent to the fiber $U = \pi^{-1}(x)$, i.e. – in terms of local coordinates – to the dependent variables $u^a \in U$ and their derivatives u_J^a , belonging (for each given J) to a vector space U_J isomorphic to U . \odot

Remark 10. The horizontal Maurer-Cartan equation (10) is automatically satisfied for Λ_i given by (13). \odot

Remark 11. Lemma 1 above deals with evolutionary representatives, which are generalized vector fields in M ; however while $X = Q^a \partial_a$ (recall here $Q^a = \varphi^a - u_k^a \xi^k$) is by construction the evolutionary representative of the Lie-point vector field $X_0 = \xi^i \partial_i + \varphi^a \partial_a$, it is not at all clear that $\tilde{X} = (A_b^a Q^b) \partial_a$ is the evolutionary representative of a Lie-point vector field \tilde{X}_0 in M . The evolutionary representatives (2) can be characterized, among first order generalized vector fields on M [51], as those satisfying

$$\partial Q^a / \partial u_k^b = -\delta_b^a \xi^k, \quad Q^a + (\partial Q^a / \partial u_k^b) u_k^b = \varphi^a;$$

note in particular that $(\partial Q^a / \partial u_k^a)$ (no sum on a) is independent of a . By applying these requests on $\tilde{X} = \hat{Q}^a \partial_a$, $\hat{Q}^a = A_b^a Q^b$, we obtain that while we always have $\hat{Q}^a + (\partial \hat{Q}^a / \partial u_k^b) u_k^b = \vartheta^a = A_b^a \varphi^b$, the first requirement is satisfied only for $A = \lambda I$, or for all the ξ^k identically vanishing. The first case amounts to a rescaling of vector fields, while the second applies when the considered symmetries do not act on independent variables. (We note in passing that examples of application of μ -symmetries seemingly always refer to either one of these cases.) \odot

We note now that the vector fields $X_\mu^{(n)}$ and $\tilde{X}_0^{(n)}$ are in general *not* collinear, and hence do *not* span the same distribution. Thus, albeit a μ -prolongation is associated to a standardly prolonged (local or nonlocal, see Section 4) vector field, these are in general not in the relation of interest here, see Section 2.

On the other hand, if $A = \lambda I$ (we will then write $X_\lambda^{(n)}$ for the twisted prolongations), then \tilde{X} is collinear to X , and $\tilde{X}_0^{(n)}$ is collinear to $X_\lambda^{(n)}$: we are then in the case where the prolonged distributions $\mathcal{D}_{\tilde{X}}^0$ and \mathcal{D}_X^λ do coincide. Recalling that in this case we deal actually with λ -prolongations and λ -symmetries, we can conclude that Lemma 1 above has a simple but relevant corollary:

Corollary 1. *Let X be a λ -symmetry for the ordinary differential equation (or system thereof) Δ . Then locally there is a (possibly nonlocal) vector field $\tilde{X} = A \circ X$ which is a standard symmetry for the same equation, with $A = \alpha I$ and the relation between α and λ is given by $\lambda = \alpha^{-1} D_x \alpha$.*

Conversely, if \tilde{X} is a standard symmetry for Δ , applying a transformation $A^{-1} = \alpha^{-1} I$ we obtain a vector field X which is a λ -symmetry for Δ , with again the same relation between α and λ .

It should be noted that twisting the prolongation operation has little effect on the set of invariant functions, and thus it is not unexpected that it also preserves the possibility of reduction of PDEs. In fact by this one usually means the search for invariant solutions, i.e. of solutions obtained by restricting the PDE to the set of X -invariant sections (i.e. X -invariant functions).

Lemma 2. *Consider the evolutionary vector fields $X = Q^a \partial_a$ and $\tilde{X} = (A^a_b Q^b) \partial_a$ on M , with A as in Lemma 1. Let \mathcal{F}_X denote the set of X -invariant sections in $\Sigma(M)$ (i.e. of X -invariant functions $f : B \rightarrow U$). Then $X_\mu^{(n)}$, $\tilde{X}^{(n)}$ and $X_0^{(n)}$ coincide on \mathcal{F}_X .*

Proof. First of all we note that invariant sections γ are characterized precisely by the fact that all the Q^a vanish on them. Thus $X_0^{(2)}$ is surely null on \mathcal{F}_X .

It was shown in [28] (see Theorem 3 in there) that writing the standard and μ -prolonged vector fields of the same evolutionary vector field $X = \varphi^a \partial_a$ in the form

$$X^{(n)} = \psi_J^a \partial_J, \quad X_\mu^{(n)} = [\psi_J^a + F_J^a] \partial_a^J,$$

the difference term F_J^a satisfies the recursion relation

$$F_{J,i}^a = (\delta_b^a D_i + (\Lambda_i)^a_b) F_J^b + (\Lambda_i)^a_b D_J Q^b.$$

This is started by $F_0^a = 0$. It is thus clear that F_J^a vanishes on \mathcal{F}_X for all J , hence $X_\mu^{(n)} = X_0^{(n)}$ on \mathcal{F}_X .

To conclude the proof, it suffices to note that the characteristics Q of X and \tilde{Q} of \tilde{X} are related by a linear (point-dependent) invertible transformation $\tilde{Q}^a = A^a_b Q^b$, hence they vanish on the same set. \triangle

Recalling point (2) in the discussion of Section 2, we immediately conclude that if an equation (or system thereof) Δ admits a vector field X as a μ -symmetry, this can be used exactly as standard symmetries in the determination of special (invariant) solutions.

Actually Lemma 2 allows to be more specific and reduce the situation to the familiar one (albeit in practice it may be more convenient to avoid this step). In fact, we have again a simple but useful corollary.

Corollary 2. *Let the equation (or system thereof) Δ admit X as a μ symmetry, with $\mu = \Lambda_i dx^i$; and let X -invariant solutions exist for Δ . Then there is a vector field $\tilde{X} = A \circ X$ which is a conditional symmetry for Δ , and the X -invariant solutions are also \tilde{X} -invariant. The relation between μ and A is given by (12).*

This Corollary allows to reduce the search for special solutions associated to μ -symmetries to the familiar case of special solution associated to standard conditional symmetries [36].

3.2 Sets of vector fields

When we consider a set \mathcal{X} of vector fields X_α , we can of course apply to each one of them the considerations presented in the previous subsection. However, in this case the generators of the distribution $\mathcal{D}_\mathcal{X}$ can also be changed by “mixing” the prolongations of the different vector fields. This is the case we are considering now, restricting to the case of ODEs (one independent variable, denoted as x).

Lemma 3. *Let $\mathcal{X} = \{X_1, \dots, X_r\}$ be a set of vector fields on M ; and let the vector fields $\mathcal{Y} = \{Y_1, \dots, Y_r\}$ on $J^n M$ be their σ -prolongation. Consider also the set $\mathcal{W} = \{W_1, \dots, W_r\}$ of vector fields on M given by $W_\alpha = A_\alpha^\beta X_\beta$, with A a nowhere singular matrix function on M ; and let the vector fields $\mathcal{Z} = \{Z_1, \dots, Z_r\}$ on $J^n M$ be their standard prolongation. Then, provided A and σ are related by*

$$\sigma = A^{-1} D_x A, \quad (15)$$

we also have $Z_\alpha = A_\alpha^\beta Y_\beta$.

Proof. It will suffice to consider first prolongations. In coordinates and with the usual shorthand notation,

$$\begin{aligned} X_\alpha &= \xi_\alpha \partial_x + \varphi_\alpha^a \partial_a, \\ Y_\alpha &= X_\alpha + ((D_x \varphi_\alpha^a - u_x^a D_x \xi_\alpha) + \sigma_\alpha^\beta (\varphi_\beta^a - u_x^a \xi_\beta)) \partial_a^1; \\ W_\alpha &= \chi_\alpha \partial_x + \eta_\alpha^a \partial_a, \\ Z_\alpha &= W_\alpha + (D_x \eta_\alpha^a - u_x^a D_x \chi_\alpha) \partial_a^1. \end{aligned}$$

If now we require $W = AX$, i.e.

$$\chi_\alpha = A_\alpha^\beta \xi_\beta; \quad \eta_\alpha^a = A_\alpha^\beta \varphi_\beta^a,$$

we immediately get

$$\begin{aligned} D_x \chi_\alpha &= (D_x A_\alpha^\beta) \xi_\beta + A_\alpha^\beta (D_x \xi_\beta), \\ D_x \eta_\alpha^a &= (D_x A_\alpha^\beta) \varphi_\beta^a + A_\alpha^\beta (D_x \varphi_\beta^a). \end{aligned}$$

Inserting these in the expression for Z , we get

$$\begin{aligned} Z &= A_\alpha^\beta [\xi_\beta \partial_x + \varphi_\beta^a \partial_a \\ &\quad + ((D_x \varphi_\beta^a - u_x^a D_x \xi_\beta) + (A^1 D_x A)_\alpha^\beta (\varphi_\beta^a - u_x^a \xi_\beta))] \partial_a^1 \\ &= A_\alpha^\beta Y_\beta + [(D_x A)_\alpha^\beta - (A\sigma)_\alpha^\beta] (\varphi_\beta^a - u_x^a \xi_\beta) \partial_a^1. \end{aligned}$$

Thus, provided A and σ satisfy $D_x A = A\sigma$, and hence (recalling A is nowhere singular) satisfy (15), we have that $W = AX$ leads to $Y = AZ$. \triangle

This Lemma shows that the distributions in $TJ^n M$ generated by \mathcal{Y} and by \mathcal{Z} do coincide. We have then immediately an analogue of Corollary 1 for the case of systems of vector fields.

Corollary 3. *Let the set $\{X_i\}$ of vector fields be a σ -symmetry for the differential equation (or system thereof) Δ . Then locally there is a set $\{W_i\}$, with $W_i = A_i^j X_j$ of vector fields which is a set of standard symmetries for the same equation; the relation between A and σ is given by (15).*

Conversely, if $\{W_i\}$ is an involution system of standard symmetries for Δ , applying a transformation A^{-1} we obtain a set $\{X_i\}$ of vector fields which is a σ -symmetry for Δ , with again the same relation between A and σ .

Remark 12. The relations between the vector fields X_i and W_i , and their (respectively, σ and standard) prolongations, as given by Lemma 3, can be summarized in the form of a commutative diagram:

$$\begin{array}{ccc} \{X_i\} & \xrightarrow{A} & \{W_i\} \\ \downarrow \sigma\text{-}prol & & \downarrow prol \\ \{Y_i\} & \xrightarrow{A} & \{Z_i\} \end{array} \quad (16)$$

The relation between A and σ is given by (15). \odot

Remark 13. As in the case of μ -prolongations, the diagram can also be read the other way round: considering the map $L = A^{-1}$ we can say that to any involution set $\{W_i\}$ of standard symmetries of an equation Δ is associated a set $\{X_i\}$, with $X_i = L_i^j W_j$, which is a σ -symmetry of the same equation Δ , the relation between $A = L^{-1}$ and σ being given by (15). \odot

Remark 14. Lemma 3 stipulates that the relation between A and σ is given by (15); note that if we look at this as an equation for A with a given σ , the solution is in general not unique (an explicit example is provided in [17]). Note also that the sets $\{Y_i\}$ and $\{Z_i\}$ considered in Lemma 3 will in general have different involution properties. A detailed discussion of this point is given in [17, 18]. \odot

Remark 15. As mentioned above, if we consider a single vector field, the matrix σ reduces to a scalar function, and the notion of σ -prolongations and symmetries reduce to that of λ -prolongations and symmetries. \odot

Remark 16. The notion of σ -symmetry can be put in contact with the "side conditions" approach by Olver and Rosenau [53, 54]; on the other hand, the latter was used by Broadbridge, Chanu and Miller [5] to study non-regular separation of variables. It is thus natural to wonder if σ -symmetries could be used to study the latter problem in general. I thank prof. Miller for pointing out the relation of σ -symmetries to his work. \odot

4 Local versus nonlocal equivalence

Our Lemma 1 and Lemma 3 above state that given a map A applied on vector fields and their prolongations, we obtain new vector fields and, in general, their twisted prolongations. This relation is summarized in the diagrams (14) and (16), see Remarks 7 and 12; the relation between A and μ or, respectively, σ is given by (13) and (15).

It should be noted, however, that these Lemmas consider the situation where one starts from a given A , obtaining the corresponding Λ_i or, respectively, σ . On the other hand, equations (13) and (15) suggest one could consider the converse problem; that is, given a set of matrices Λ_i , or a matrix σ , wonder if there is an associated map defined by A . The purpose of this section is indeed to briefly discuss this point, i.e. the relations between the “direct” and the “inverse” problem.

Note that while for a given function $A = A(x, u)$, eq.(13) obviously defines smooth local functions $\Lambda_i = \Lambda_i(x, u, u_x)$, if we look for the A which satisfy (13) for given Λ_i , it is much less obvious that a solution exists, and even less that A will be a local function of (x, u) . The first fact is guaranteed by the (horizontal) Maurer-Cartan equation (10), but the second is in general not true.

In fact, the solution to (13) is written formally as

$$A = \exp \left[\int \Lambda_i \cdot dx^i \right] ; \quad (17)$$

unless the Λ_i are themselves obtained as total derivatives of some “pseudo-potential”, $\Lambda_i = D_i \Phi$, this will *not* be a local function of the x and u .

This point will be made clearer by an explicit example. For two independent variables (x, y) and one dependent variable u , consider

$$\Lambda_x = \begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix} ;$$

one can check that the most general form of Λ_y satisfying the horizontal Maurer-Cartan equation (10) is

$$\Lambda_y = \begin{pmatrix} f_1(x, y) - u f_2(x, y) & u_y - u^2 f_2(x, y) + u[f_1(x, y) - f_3(x, y)] + f_4(x, y) \\ f_2(x, y) & f_3(x, y) + u f_2(x, y) \end{pmatrix},$$

where the f_i are arbitrary smooth functions of their argument. If we choose e.g.

$$\Lambda_x = \begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix}, \quad \Lambda_y = \begin{pmatrix} 0 & u_y + h'(y) \\ 0 & 0 \end{pmatrix} ;$$

these satisfy (10), and the A identified by (13) is simply

$$A = \begin{pmatrix} c_1[u + h(y)] & c_2 + c_3[u + h(y)] \\ c_1 & c_3 \end{pmatrix}.$$

On the other hand, if we require that all the functions f_i are nowhere zero, e.g. set all them to be nonzero constants ($f_i x, y = k_i \neq 0$), we get

$$\Lambda_y = \begin{pmatrix} k_1 - k_2 u & -k_2 u - k_2 u^2 + u_y + k_4 \\ k_2 & k_1 + k_3 + k_2 u \end{pmatrix} ;$$

it is then easy to satisfy (13) for Λ_x with a local function $A = A(x, y, u)$; this is for any A of the form

$$A = \begin{pmatrix} \alpha_1(y) + u\alpha_2(y) & \alpha_3(y) + u\alpha_4(y) \\ \alpha_2(y) & \alpha_4(y) \end{pmatrix} .$$

But when we try to satisfy $D_y A = \Lambda_y A$ for A in this class, the only possibility is given by $A = 0$; that is, there is no solution to (13) with local functions.

Remark 17. It should be noted that in a previous paper of mine and coauthors [16], some result by Marvan [38] was incorrectly translated from the original mathematical language of coverings and bi-complexes to the language used in that paper. As a result, our paper gave a proposition (Proposition 2 in there) attributed to Marvan which is not equivalent to what was proved by Marvan in [38] and which appears to be incorrect, as shown above. \odot

Remark 18. Note also that even in the ODE case, when only one x and one Λ are present and hence no Maurer-Cartan condition is applied, the A can very well be a non-local function: this is e.g. definitely the case when $\Lambda = \Lambda(u)$. This phenomenon is well known, and the equivalence of λ -symmetries with non-local symmetries has been studied in the literature [7, 8, 9, 44, 47, 49]. \odot

Remark 19. Note in this respect that in [24] and [26] one considered gauge transformations that were obtained by evaluating a certain G -valued function (with G the gauge group) on a section of the gauge bundle over M ; this amounted to a gauge fixing. Thus in this case one reduces to evaluation of functions or integrals over a specific section. \odot

5 Examples

We present some simple examples illustrating our discussion in concrete cases.

Example 1 (λ -symmetries). In their seminal paper [40] Muriel & Romero, following the discussion by Olver in his book [51], considered equations of the form

$$u_{xx} = D_x F(x, u) . \quad (18)$$

These can obviously be integrated by quadratures, but may lack symmetries; this is e.g. the case for

$$F(x, u) = (x + x^2) e^u . \quad (19)$$

It was proven by Muriel & Romero (see Theorem 4.1 in [40]) that any equation of the form (18) admits the vector field $X = \partial/\partial u$ as a λ -symmetry with

the choice $\lambda = F_u(x, u)$; this allows to integrate the equation via (λ) symmetry reduction. Thus in particular with the choice (19), i.e. for the equation

$$u_{xx} = [1 + 2x + u_x(x + x^2)] e^u ,$$

we have $\partial/\partial u$ as a λ -symmetry with

$$\lambda = (x + x^2) e^u .$$

The λ -prolonged vector field (acting in J^2M , as we consider a second order equation) turns out to be

$$Y = \frac{\partial}{\partial u} + [(x+x^2) e^u] \frac{\partial}{\partial u_x} + [(e^u(x^2 + x)^2 + 2x + xu_x(x+1) + 1) e^u] \frac{\partial}{\partial u_{xx}} .$$

According to our Corollary 1, this is equivalent to a standardly prolonged vector field Z , which is the prolongation of a Lie-point vector field W ; these are related to Y and X by a function $\alpha(x, u)$ via $W = \alpha X$, $Z = \alpha Y$, and α satisfies $\lambda = \alpha^{-1}(D_x \alpha)$.

This produces a non-local function α , i.e.

$$\alpha = \exp \left[\int (x + x^2) e^{u(x)} dx \right] .$$

The vector field W will then be simply $W = \alpha \partial_u$; its standard prolongation turns out to be

$$Z = \alpha \partial_u + (D_x \alpha) \partial_{u_x} + (D_x^2 \alpha) \partial_{u_{xx}} ;$$

using $D_x \alpha = \alpha \lambda$ we get

$$Z = \alpha \partial_u + \alpha \lambda \partial_{u_x} + \alpha (\lambda^2 + D_x \lambda) \partial_{u_{xx}} = \alpha Y ,$$

which confirms our general result. In this case, due to the factor α , Z is of course a non-local vector field (of exponential type) [51].

Example 2 (μ -symmetries). Let us now consider an example of μ -prolongation for ODEs, hence with a single matrix Λ , keeping to the two-dimensional and second order case for the sake of simplicity; we denote the independent variable by x , the dependent ones by (u, v) .

We choose Λ as follows; the associated A is then given by (12):

$$\Lambda = \begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix} ; \quad A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} . \quad (20)$$

Let us now consider the vector field

$$X = \partial/\partial v ;$$

its second order μ -prolongation turns out to be

$$Y = X_\mu^{(2)} = \frac{\partial}{\partial v} + u_x \frac{\partial}{\partial u_x} + u_{xx} \frac{\partial}{\partial u_{xx}} .$$

Differential invariants of order up to two for this vector field are (note that, as mentioned in our general discussion, the IBDP does not hold):

$$x, u; u_x e^{-v}, v_x; u_{xx} e^{-v}, v_{xx} .$$

Thus X is a μ -symmetry for e.g. all the equations in the class

$$\begin{aligned} u_{xx} &= f_{11}(x, u) u_x + f_{12}(x, u) e^v \\ v_{xx} &= f_{21}(x, u) v_x + f_{22}(x, u) , \end{aligned}$$

with $f_{ij}(x, u)$ arbitrary smooth functions.

According to our Lemma 1, the vector field $Z = A \circ X_\mu^{(2)}$ should be the standard prolongation of the vector field $W = A \circ X$. This is indeed the case, as W and Z are given respectively by

$$W = u \frac{\partial}{\partial u} + \frac{\partial}{\partial v} ; \quad Z = W + u_x \frac{\partial}{\partial u_x} + u_{xx} \frac{\partial}{\partial u_{xx}} .$$

We note in passing that the differential invariants of order up to two for Z are

$$x, u e^{-v}; u_x/u, v_x; u_{xx}/u, v_{xx} ;$$

thus the IBDP holds for Z . Note that the differential invariants for the vector fields Y and Z are not the same.

Example 3 (σ -symmetries). In order to provide a simple example of σ -symmetries, again with one independent variable x and two dependent variables u, v , we consider the scaling and rotation vector fields given by

$$X_1 = u \partial_u + v \partial_v , \quad X_2 = v \partial_u - u \partial_v ;$$

and the matrix (which of course acts actually only on the first vector field)

$$\sigma = \begin{pmatrix} 0 & u_x \\ 0 & 0 \end{pmatrix} .$$

The second order σ -prolongation of the set $\mathcal{X} = \{X_1, X_2\}$ is provided by $\mathcal{Y} = \{Y_1, Y_2\}$ with

$$\begin{aligned} Y_1 = (X_1)_\sigma^{(2)} &= u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + u_x(1+v) \frac{\partial}{\partial u_x} + (v_x - u u_x) \frac{\partial}{\partial v_x} \\ &\quad + [u_{xx}(1+v) + 2u_x v_x] \frac{\partial}{\partial u_{xx}} + (v_{xx} - u u_{xx} - 2u_x^2) \frac{\partial}{\partial v_{xx}} , \\ Y_2 = (X_2)_\sigma^{(2)} &= v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} + v_x \frac{\partial}{\partial u_x} - u_x \frac{\partial}{\partial v_x} + v_{xx} \frac{\partial}{\partial u_{xx}} - u_{xx} \frac{\partial}{\partial v_{xx}} . \end{aligned}$$

According to Lemma 3, these should be equivalent to a set of standardly prolonged vector fields Z_i , given by $Z_i = A_i^j Y_j$; more precisely the latter should be the prolongation of the vector fields $W_i = A_i^j Y_j$, and A is determined by (15).

In this case we obtain

$$A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} ;$$

this yields at once

$$\begin{aligned} W_1 &= u(1+v) \frac{\partial}{\partial u} + (v-u^2) \frac{\partial}{\partial v} , \\ W_2 &= v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} ; \\ Z_1 &= u(1+v) \frac{\partial}{\partial u} + (v-u^2) \frac{\partial}{\partial v} + [u_x(1+v) + uv_x] \frac{\partial}{\partial u_x} + (v_x - 2uu_x) \frac{\partial}{\partial v_x} \\ &\quad + [u_{xx}(1+v) + 2u_x v_x + uv_{xx}] \frac{\partial}{\partial u_{xx}} + (v_{xx} - 2uu_{xx} - 2u_x^2) \frac{\partial}{\partial v_{xx}} , \\ Z_2 &= v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} + v_x \frac{\partial}{\partial u_x} - u_x \frac{\partial}{\partial v_x} + v_{xx} \frac{\partial}{\partial u_{xx}} - u_{xx} \frac{\partial}{\partial v_{xx}} . \end{aligned}$$

It is immediate to check that Z_i is the second standard prolongation of W_i .

Example 4 (non vertical vector fields). The examples considered so far dealt with vertical vector fields; here we present, in the context of μ -symmetries, an example where the vector field is not vertical. We will again deal with one independent (x) and two dependent (u, v) variables and second order prolongations; and use Λ – and hence A – as in Example 2 above, see (20). We consider the vector field

$$X = -\partial_x + u \partial_u + v \partial_v ; \quad (21)$$

its evolutionary representative is

$$X_v = (u + u_x) \partial_u + (v + v_x) \partial_v . \quad (22)$$

Note moreover that the X -invariant functions are written as

$$u = k_1 e^{-x} , \quad v = k_2 e^{-x} . \quad (23)$$

Applying A as in (20) on these, we obtain

$$\begin{aligned} W &= A \cdot X = -\partial_x + u(1+v) \partial_u + v \partial_v , \\ W_v &= A \cdot X_v = [u_x + u(1+v+v_x)] \partial_u + (v+v_x) \partial_v . \end{aligned}$$

The second order μ -prolongation of X and of X_v are now easily computed (we will not write down the explicit formulas, which are rather long and of no special interest); we will also denote them by $Y = X_\mu^{(2)}$ and $Y_v = (X_v)_\mu^{(2)}$. Applying A on these, we obtain $Z = A \cdot Y$ and $Z_v = A \cdot Y_v$.

Our Lemma 1, which requires to deal with vertical vector fields, states that $Z_v = W_v^{(2)}$, and this is indeed the case. On the other hand one can check that

$Z \neq W^{(2)}$; more precisely, we have $Z = W^{(2)} + u_x v_x \partial_{u_x} + u_x v_{xx} \partial_{u_{xx}}$: this shows that the equivalence does indeed hold only between the evolutionary representatives. We stress here that $W_v = A \cdot X_v$ is *not* the evolutionary representative of $W = A \cdot X$; the latter is

$$W_{ev} = [u_x + u(1 + v)] \partial_u + (v + v_x) \partial_v .$$

Let us now consider, beside Y, Y_v, Z, Z_v defined above, also the second standard prolongation of X and X_v , i.e. $X_0^{(2)}$ and $(X_v)_0^{(2)}$; these have of course no special relation with $Z, W^{(2)}$ or $Z_v = W_v^{(2)}$. According to Lemma 2, however, we should have that $Y_v = Z_v = X_v^{(2)}$ (and actually vanish) on the set of invariant functions, and this is indeed the case.

6 Conclusions

In this paper we have considered known notions and results, and provided a new way to look at them and interpretation with the aim of advancing our understanding of twisted prolongations and of their effectiveness in determining solutions to differential equations.

We have considered the classical theory of symmetries of differential equations; we have noted that what matters in establishing if a vector field (an involution set of vector fields) is a symmetry (a symmetry algebra) of a given differential equation of order n is not the prolongation of a vector field (of a set of vector fields), but the distribution generated by this (by these) in $J^n M$, so that changing the generators of the distribution does not alter the property of being a symmetry (a symmetry algebra).

We have then noted that changing the set of generators of the above mentioned distribution still requiring that they are projectable to each $J^k M$ (for $k \leq n$), and for the setting of ODEs that they satisfy the IBDP leads us to consider λ -prolongations or, in the case of sets of vector fields, σ -prolongations. On the other hand, requiring that the set of invariant functions is the same as for the original generators – which is the natural requirement on the setting of PDEs, leads us to consider μ -prolongations. For sets of vector fields, these can be combined with σ ones to produce combined twisted prolongations, i.e. χ -prolongations.

The change of generators for the distribution is naturally interpreted in terms of a gauge action. This holds both for λ and μ prolongations, in which case the gauge action is on the space of dependent variables and their variables; and in the case of σ -prolongations, in which case it is on the space of vector fields (see above for more details).

We have also stressed that while changing the set of generators produces twisted symmetries, the correspondence between the gauge action and the twisted symmetries is such that if we consider a given twisted symmetry and look for a change of generators for the distribution (this is naturally indexed by a gauge transformation) corresponding to this, we will in general obtain a nonlocal map.

We conclude that while our discussion shows that twisted symmetries can be understood in terms of change of generators for the relevant distributions of prolonged vector fields, this point of view would in general lead to consider non-local maps. In other words, twisted symmetries are actually more general than standard symmetries.

On the other hand, the general – and generally formal, see above – correspondence unveiled by the present work simply explains why twisted symmetries are as effective as standard ones in the symmetry analysis of differential equations.

Appendix A.

Prolongation of vector fields in involution

In this Appendix we establish the relation between the involution relations for vector fields in M and for their prolongation in $J^n M$.

In other words, we want to consider vector fields of the form

$$Z = f^\alpha(x, u) X_\alpha$$

and describe prolongations of Z in terms of the prolongations of the X_α , for general smooth functions f^α .

We will write the basis vector fields as $X_\alpha = \xi_\alpha^i \partial_i + \varphi_\alpha^a \partial_a$, and

$$Z = f^\alpha X_\alpha = \Xi^i \partial_i + \Phi^a \partial_a ; \quad (24)$$

needless to say, we have $\Xi^i = f^\alpha \xi_\alpha^i$, $\Phi^a = f^\alpha \varphi_\alpha^a$.

We will write the prolonged field as

$$Z^{(n)} = \Xi^i \partial_i + \chi_J^a \partial_a^J \quad (25)$$

(sum over J is meant with $|J| \leq n$), with $\chi_0^a = \Phi^a$. We like to write this prolongation as

$$Z^{(n)} = f^\alpha X_\alpha^{(n)} + \Gamma^{(n)} . \quad (26)$$

In terms of the coefficients of the vector field $Z^{(n)}$, this means

$$\chi_J^a = f^\alpha \psi_{\alpha;J}^a + W_J^a , \quad (27)$$

where $\psi_{\alpha;J}^a$ are the coefficients in the (standard) prolongation of X_α .

We will also use the notation

$$Q_{\alpha;J}^a := \psi_{\alpha;J}^a - u_{J,k}^a \xi_\alpha^k . \quad (28)$$

Lemma A.1. *The coefficients W_J^a introduced above satisfy the recursion relation*

$$W_{J,i}^a = (D_i W_J^a) + (D_i f^\alpha) Q_{\alpha;J}^a , \quad (29)$$

where $W_0^a = 0$.

Proof. Applying the prolongation formula, we have

$$\chi_{J,i}^a = D_i \chi_J^a - (u_{J,k}^a D_i \Xi^k) ;$$

recalling now the definition of the Ξ , see (24) and the above expression (27) for the χ , this yields

$$\begin{aligned} \chi_{J,i}^a &= D_i (f^\alpha \psi_{\alpha;J}^a + W_J^a) - u_{J,k}^a D_i (f^\alpha \xi_\alpha^k) \\ &= (D_i f^\alpha) [\psi_{\alpha;J}^a - u_{J,k}^a \xi_\alpha^k] + (D_i W_J^a) + f^\alpha (D_i \psi_{\alpha;J}^a - u_J^a D_i \xi_\alpha^k) \\ &= \psi_{\alpha;J,i}^a + (D_i W_J^a) + (D_i f^\alpha) [\psi_{\alpha;J}^a - u_{J,k}^a \xi_\alpha^k] . \end{aligned}$$

Using the notation (28), and going back to the expression (27), the recursion relation for the W results to be (29). \triangle

This Lemma allows to give the generalization of the standard result about prolongations of Lie algebras of vector fields to the case of vector fields in involution.

Lemma A.2. *Let the vector fields X_α ($\alpha = 1, \dots, r$) be in involution,*

$$[X_\alpha, X_\beta] = F_{\alpha\beta}^\gamma(x, u) X_\gamma .$$

Then their prolongations satisfy

$$[X_\alpha^{(n)}, X_\beta^{(n)}] = F_{\alpha\beta}^\gamma X_\gamma^{(n)} + \Gamma_{\alpha\beta}^{(n)} , \quad (30)$$

where the coefficients $W_{\alpha\beta;J}^a$ of the vector fields $\Gamma_{\alpha\beta}^{(n)} = W_{\alpha\beta;J}^a \partial_a^J$ satisfy $W_{\alpha\beta;0}^a = 0$ and the recursion relations

$$W_{\alpha\beta;J,i}^a = (D_i W_{\alpha\beta;J}^a) + (D_i F_{\alpha\beta}^\gamma) Q_{\gamma;J}^a , \quad (31)$$

with $W_{\alpha\beta;0}^a = 0$.

Proof. Basically this follows by Lemma A.1, introducing the two indices α and β in our previous discussion. We know by Lemma 1 that

$$[X_\alpha^{(n)}, X_\beta^{(n)}] = \left(F_{\alpha\beta}^\gamma(x, u) X_\gamma \right)^{(n)} := Z_{\alpha\beta}^{(n)} .$$

The vector field $Z_{\alpha\beta}$ on the r.h.s. is of the form considered in the discussion leading to Lemma A.1; thus we know that

$$Z_{\alpha\beta}^{(n)} = F_{\alpha\beta}^\gamma X_\gamma^{(n)} + \Gamma_{\alpha\beta}^{(n)}$$

where the vector fields $\Gamma_{\alpha\beta}^{(n)}$ can be written as $\Gamma_{\alpha\beta}^{(n)} = W_{\alpha\beta;J}^a \partial_a^J$. Again by Lemma A.1, the coefficients $W_{\alpha\beta;J}^a$ satisfy (31). \triangle

Appendix B.

Equivalent symmetries

Our discussion led us to consider deformations of the prolongation operation. It is thus advisable, when considering symmetries of a differential equation (say of order n), to keep track not only of the vector field X in M , but also of the prolongation operation \mathcal{P} used to generate from X a vector field in $J^n M$. In this context we will denote by (X, \mathcal{P}) the pair consisting of a vector field and a prolongation operation; we will denote by \mathcal{P}_0 the standard prolongation operation. When $\mathcal{P} \neq \mathcal{P}_0$ we have a twisted prolongation. We will also denote by $X_{\mathcal{P}}^{(n)}$ the vector field in $J^n M$ obtained by prolonging X via the \mathcal{P} prolongation operation.

We can then generalize Definition 2 by the following (which reduces to Definition 2 for $\mathcal{P} = \mathcal{P}_0$)

Definition B.1. *The vector field X in M with the prolongation operation \mathcal{P} is a \mathcal{P} -symmetry generator for the differential equation Δ of order n if $S_{\Delta} \subset J^n M$ is an integral manifold for the distribution \mathcal{D}_X spanned by $X_{\mathcal{P}}^{(n)}$ in $J^n M$. The set $\mathcal{X} = \{X_1, \dots, X_r\}$ of vector fields in M with the prolongation operation \mathcal{P} is a set of \mathcal{P} -symmetry generators for the differential equation Δ of order n if $S_{\Delta} \subset J^n M$ is an integral manifold for the distribution $\mathcal{D}_{\mathcal{X}}^{(\mathcal{P})}$ spanned by $X_{\alpha, \mathcal{P}}^{(n)}$ in $J^n M$.*

When we want to keep track of the prolongation used, we will denote the distribution by $\mathcal{D}_X^{\mathcal{P}}$, or the like for sets of vector fields.

As we consider general prolongation operations, it makes sense to consider vector fields which are prolonged through *different* prolongation operations to produce the *same* distribution in $J^n M$.

Definition B.2. *Assume the vector fields X and Y in M are respectively a \mathcal{P} -symmetry and a \mathcal{P}' -symmetry generator of the differential equation Δ of order n , with \mathcal{P} and \mathcal{P}' possibly different. We say that (X, \mathcal{P}) and (Y, \mathcal{P}') are equivalent symmetries of Δ if they give raise to the same distribution in $J^n M$, i.e. if $\mathcal{D}_X = \mathcal{D}_Y$. The sets of vector fields \mathcal{X} and \mathcal{Y} in M , both symmetry sets of the differential equation Δ of order n , are equivalent sets of symmetries of Δ if they span the same distribution in $J^n M$, i.e. if $\mathcal{D}_{\mathcal{X}}^{(\mathcal{P})} = \mathcal{D}_{\mathcal{Y}}^{(\mathcal{P}')}$.*

We stress that in the case of sets, equivalent sets are not required to have the same cardinality: indeed, we have not required that the X_{α} (and the Y_{α}), nor the $X_{\alpha, \mathcal{P}}^{(n)}$ (and the $Y_{\alpha, \mathcal{P}'}^{(n)}$) be independent at each point, and not even that they are sets of independent vector fields.

The results of Section 3 can be restated in terms of equivalent symmetries. In particular, Corollary 1 guarantees that $(X, \mathcal{P}_{\lambda})$ and $(\tilde{X}, \mathcal{P}_0)$ are equivalent symmetries; similarly, Corollary 3 guarantees that the sets $(\mathcal{X}, \mathcal{P}_{\sigma})$ and $(\mathcal{W}, \mathcal{P}_0)$ are equivalent sets.

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